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On the KdV-type equation with variable coefficients

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Abstract. In this paper, the KdV-type equations with variable coefficients appearing in a lot of the literature are discussed generally. The explicit transformations which transform various KdV, *m*KdV and KP-type nonlinear evolution equations into their 'canonical' forms with constant coefficients are given. The results presented here tie up many of the investigations scattered in the literature.

1. Introduction

In the last decade there has been intense activity regarding the study of the complete integrability and symmetries of nonlinear partial differential equations (PDEs). But the main domain of activity was generally restricted to equations with constant coefficients. Recently, nonlinear PDEs with variable coefficients have been discussed by various authors. If one searches a variety of literature, one really is surprised how much work has been invested in the study of the KdV-type equations with variable coefficients which originated, from shallow water problems in water with variable depth and inhomogeneous properties of media. It has been shown that x, t-dependent KdV-type equations play an important role in applications. In this paper, KdV-type equations with variable coefficients which appear in a lot of the literature are discussed generally. The explicit transformations which transform various KdV-, *m*KdV- and KP-type nonlinear evolution equations into their 'canonical' forms with constant coefficients are given. The results presented here tie up many of the investigations scattered in the literature.

2. Transformations from the general KdV-type equations with variable coefficients to KdV-type equations

We consider the general KdV-type equations with variable coefficients

$$u_t = f_3(t)u_{xxx} + 6f_2(t)u^p u_x + f_0(t)u + f_1(t)xu_x + s(t)u_x \qquad p > 0. \tag{1}$$

Several equations of physical interest are covered by equation (1) (see below for examples). Let transformations

$$u(x,t) = c_2 g(t)V(\zeta,\tau) \qquad \zeta = xh(t) + l(t) \qquad \tau = m(t) \tag{2}$$

where

$$g(t) = c_0 \exp\left(\int^t f_0(t)\,\mathrm{d}t\right) \qquad h(t) = c_1 \exp\left(\int^t f_1(t)\,\mathrm{d}t\right) \tag{3a}$$

$$l(t) = \int^t s(t)h(t)\,\mathrm{d}t \qquad m(t) = \int^t f_3(t)h^3(t)\,\mathrm{d}t \tag{3b}$$

c_0, c_1 and c_2 are arbitrary constants. If coefficients $f_i(t)$ satisfy the equation

$$f_3(t)h^2(t) = c_2 f_2(t) g^p(t) \tag{4}$$

then equation (1) is transferred into a KdV-type equation

$$V_\tau = V_{\zeta\zeta\zeta} + 6V^p V_\zeta. \tag{5}$$

If $p = 1$, coefficients $f_i(t)$ do not satisfy equation (4), then let transformation

$$\begin{aligned} &u(x,t) = \psi(t) + c_3\varphi(t)x + c_2 g(t)V(\zeta,\tau), \varphi(t) \neq 0 \\ &\zeta = xh(t) + l(t) \qquad \tau = m(t) \end{aligned} \tag{6}$$

where

$$\begin{aligned} &\psi'(t) = (6c_3 f_2(t)\varphi(t) + f_0(t))\psi(t) + c_3 s(t)\varphi(t) \\ &l(t) = \int^t (s(t)h(t) + 6f_2(t)\psi(t)h(t))\,dt \end{aligned} \tag{7a}$$

$$\begin{aligned} &\varphi'(t) = 6c_3 f_2(t)\varphi^2(t) + (f_0(t) + f_1(t))\varphi(t) \\ &g(t) = c_0 \exp\left(\int^t (6c_3 f_2(t)\varphi(t) + f_0(t))\,dt\right) \end{aligned} \tag{7b}$$

$$\begin{aligned} &h(t) = c_1 \exp\left(\int^t (6c_3 f_2(t)\varphi(t) + f_1(t))\,dt\right) \\ &m(t) = \int^t f_3(t)h^3(t)\,dt \end{aligned} \tag{7c}$$

c_0, c_1, c_2 and c_3 are arbitrary constants. If coefficients $f_i(t)$ satisfy the equation

$$f_3(t)h^2(t) = c_2 f_2(t) g(t) \tag{8}$$

then equation (1) ($p = 1$) reads as the KdV. So, equation (1) is an essentially constant coefficient KdV-type. The KdV-type equations with variable coefficients appearing in a lot of the literature are special cases of equation (1). Let us consider several examples.

Example 1. Equation (5) admits the solitary wave solution

$$V(\zeta,\tau) = A\,\mathrm{sech}^{2/p} k\eta + B\delta_{p1} \qquad \eta = \zeta - c\tau \qquad \delta_{p1} = \begin{cases} 1 & p = 1 \\ 0 & p \neq 1 \end{cases} \tag{9}$$

where

$$A = \left(\frac{(p+1)(p+2)k^2}{3p^2}\right)^{1/p} \qquad c = -\frac{4k^2}{p^2} - 6B\delta_{p1}. \tag{10}$$

We thus get the solitary wave solution to equation (1)

$$\begin{aligned} u(x,t) = c_0 c_2 \exp\left(\int^t f_0(t)\,dt\right)\Bigg\{ A\,\mathrm{sech}^{2/p}\Bigg[k\Bigg(c_1 x \exp\left(\int^t f_1(t)\,dt\right) \\ + c_1 \int^t s(t) \exp\left(\int^t f_1\,dt\right) - c c_1^3 \int^t f_3 \exp\left(3\int^t f_1\,dt\right) dt\Bigg)\Bigg] + B\delta_{pl}\Bigg\} \end{aligned} \tag{11}$$

when $f_i(t)$ satisfies equation (4). Equation (1) ($p = 1$) also admits the solution,

$$\begin{aligned} u(x,t) = \psi(t) + c_3\varphi(t)x + c_2 c_0 \exp\left(\int^t (6c_3 f_2\varphi + f_0)\,dt\right) \\ \times(2k^2\,\mathrm{sech}^2\{k[xh(t) + l(t) + (4k^2 + 6B)m(t)]\} + B) \end{aligned} \tag{12}$$

where $\psi(t)$, $\varphi(t)$, $h(t)$, $l(t)$ and $m(t)$ are presented by equations (7*a*–*c*), if $f_i(t)$ satisfies equation (8).

Example 2. Evidently, equation

$$u_t = \nu(t)^3 T_t(t) u_{xxx} + 6\nu(t)\omega(t)^p T_t(t) u^p u_x - \frac{\omega_t(t)}{\omega(t)} u - \frac{\nu_t(t)}{\nu(t)} x u_x + s(t) u_x$$
$$T_t(t) = \frac{dT(t)}{dt} \tag{13}$$

is a special case of equation (1). Equation (13) is transformed into equation (5) through the transformation

$$u = c_2 c_0 \omega(t)^{-1} V\left(c_1 x \nu(t)^{-1} + c_1^3 \int^t \frac{s(t)}{\nu(t)} dt, c_1^3 T(t)\right). \tag{14}$$

Let $p = 1$ and $s(t) = 0$, equation (13) reads as the equation derived by Fuchssteiner [1]. Pseudopotentials, Lax pairs, Backlund transformation (BT), infinite conservation laws (ICL), symmetry and Lie algebra for the following equation

$$u_t = b(t)(u_{xxx} + 6uu_x) + [F(t)x + G(t)]u_x + 2F(t)u \tag{15}$$

are obtained (see [2–4]). Evidently, equation (15) is also a special case of equation (1), equation (15) reduces to the KdV by means of the following transformation

$$u(x,t) = \exp\left(2\int^t F(t)\,dt\right) V\left(x \exp\left(\int^t F(t)\,dt\right) + \int^t G(t) \exp\left(\int^t F(t)\,dt\right) dt, \int^t b(t) \exp\left(3\int^t F(t)\,dt\right) dt\right). \tag{16}$$

Example 3. Equation (1) contains

$$u_t = T_t(t)\nu(t)^3 u_{xxx} + 6\omega(t)\nu(t)T_t(t) u u_x - \psi_t(t) \ln \frac{a(t)}{a(T(t))} u - \psi_t(t) \ln \frac{b(t)}{(b(T(t))} x u_x \tag{17}$$

where

$$\omega(t) = \exp\left\{\int^t \psi_s(s) \ln\left[\frac{a(s)}{a(T(s))}\right] ds\right\} \qquad \nu(t) = \exp\left\{\int^t \psi_s(s) \ln\left[\frac{b(s)}{b(T(s))}\right] ds\right\} \tag{18}$$

an equation introduced by Fuchssteiner (see [1]). Notice that

$$\frac{\omega_t}{\omega} = \psi_t(t) \ln \frac{a(t)}{a(T(t))} \qquad \frac{\nu_t}{\nu} = \psi_t(t) \ln \frac{b(t)}{b(T(t))} \tag{19}$$

so equation (17) is transformed into the KdV through the transformation

$$u(x,t) = \omega(t)^{-1} V(x\nu(t)^{-1}, T(t)). \tag{20}$$

It should be remarked that equations (3.71) and (3.72) of [1] contain some errors, should be modified to equations (17), (18) and (20) above.

Example 4. Lax pairs, BT and the Painlevé property (PP) of variable coefficient KdV

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0 \qquad m = n \text{ or } m = 2n+1 \qquad \alpha\beta = \text{constants} \tag{21}$$

are obtained by Nirmale *et al* [5]. Equation (21) is also covered by equation (1). Evidently, if $m = n$, then equation (21) is reduced to the KdV. If $m \neq n$, perform the following transformations

$$u = \frac{6\beta t^{m-n}}{\alpha}[c_1 x \varphi(t) + g(t) V(\zeta, \tau)] \tag{22a}$$

$$\zeta = x h(t) \qquad \tau = m(t) = -\int^t \beta t^m h^3(t)\,dt \tag{22b}$$

where

$$\varphi'(t) = -6c_1\beta t^m \varphi^2(t) - \frac{m-n}{t}\varphi(t) \qquad h(t) = \exp\left(-\int 6c_1\beta t^m \varphi \, \mathrm{d}t\right) \tag{22c}$$

$$g(t) = \exp\left[-\int \left(6c_1\beta t^m \varphi + \frac{m-n}{t}\right) \mathrm{d}t\right]. \tag{22d}$$

Choose $c_1 = (n+1)/6\beta$, then $\varphi(t) = g(t) = 1/t^{m+1}$, $h(t) = 1/t^{n+1}$. So, if $m = 2n+1$, then equation (21) reduces to the KdV, through transformation

$$u(x,t) = \frac{(n+1)x}{\alpha t^{n+1}} + \frac{6\beta}{\alpha t^{n+1}} V\left(\frac{x}{t^{n+1}}, \frac{\beta}{(n+1)t^{n+1}}\right). \tag{23}$$

Example 5. The equation

$$u_t = \frac{\alpha}{t}(u_{xxx} + 6uu_x) + \frac{\beta}{t}u + \frac{\gamma}{t}xu_x \qquad \alpha, \beta, \gamma = \text{constants} \tag{24}$$

is a special case of equation (1). We find that equation (24) can be reduced to the KdV if $\gamma = 2\beta$, by means of the transformation

$$u(x,t) = -\frac{\beta}{2\alpha}x + t^{-2\beta} V\left(xt^{-\beta}, -\frac{\alpha}{3\beta}t^{-3\beta}\right). \tag{25}$$

If $\alpha = 1$, $\beta = -\frac{1}{3}$, we obtain equations (3.79) and (3.80) of [1].

Example 6. It is demonstrated that the KdV equation with non-uniformities

$$u_t + a(t)u + [b(t,x)u]_x + c(t)uu_x + d(t)u_{xxx} + e(x,t) = 0 \tag{26}$$

has the PP, BT and Lax pairs if the coefficients satisfy the compatibility condition

$$\begin{aligned} b_t + (a - Lc)b + bb_x + d(t)b_{xxx} &= 2ah + hL\frac{d(t)}{c^2} + \frac{\mathrm{d}h}{\mathrm{d}t} + ce \\ &\quad + x\left(2a^2 + aL\frac{d^3(t)}{c^4} + \frac{\mathrm{d}a}{\mathrm{d}t} + L\frac{d(t)}{c}L\frac{d(t)}{c^2} + \frac{\mathrm{d}}{\mathrm{d}t}L\frac{d(t)}{c}\right) \end{aligned} \tag{27}$$

where $L = (\mathrm{d}/\mathrm{d}t)\ln$, and $h(t)$ is an arbitrary and sufficiently smooth function of t [6]. Evidently, the KdV equation for non-uniform media with relaxation effects [7]

$$u_t + ru + [(c_0 + ru)u]_x + 6uu_x + u_{xxx} = 0 \qquad r = \text{constant} \tag{28}$$

the cylindrical KdV equation [8]

$$u_t + \frac{u}{2t} + 6uu_x + u_{xxx} = 0 \tag{29}$$

and the equation with non-uniform terms [9]

$$u_t + ru + 6uu_x + u_{xxx} = \tfrac{1}{3}r^2 x \qquad r = \text{constant} \tag{30}$$

are special cases of equation (26). Let transformation $u(x,t) = (1/c(t))[6d(t)w(x,t) - b(x,t)]$, by using the compatibility condition (27), equation (26) is reduced to the equation

$$w_t = -d(t)(6ww_x + w_{xxx}) - 6d(t)f(t)w + x[f'(t) + 12d(t)f^2(t)] + F(t) \tag{31}$$

where

$$f(t) = \frac{a + L(d/c)}{6d} \qquad F(t) = \frac{1}{6d}\left(2ah + hL\frac{d}{c^2} + \frac{\mathrm{d}h}{\mathrm{d}t}\right). \tag{32}$$

The PP, Lax pairs, BT, symmetries, Lie algebra and soliton-like solution for equation (31) are discussed by Lou Sen-yue [10], Tian Chou [11], and Zhu Zuo-nong [12]. Putting $w(x,t) = U(x,t) + f(t)x + f_0(t)$, then equation (31) becomes

$$U_t = -d(t)(6UU_x + U_{xxx}) - 6d(t)f(t)(xU_x + 2U) + 6f_0(t)U_x \tag{33}$$

with

$$f_0(t) = \exp\left(-\int^t 12d(t)f(t)\,\mathrm{d}t\right)\left[\int^t F(t)\exp\left(12\int^t d(t)f(t)\,\mathrm{d}t\right) + c_0\right] \tag{34}$$

$c_0 = \text{constant}$.

Equation (33) coincides with equation (1) ($p = 1$). So equation (26) is reduced to the KdV by using the transformation

$$\begin{aligned} u(x,t) = \frac{1}{c(t)}\bigg[&6d(t)f_0(t) - b(x,t) + \left(a + L\frac{d}{c}\right)x\bigg] \\ &+\frac{6d(t)}{c(t)}\exp\left(-2\int^t\left(a + L\frac{d}{c}\right)\mathrm{d}t\right] \\ &\times V\bigg(x\exp\left[-\int^t\left(a + L\frac{d}{c}\right)\mathrm{d}t\right] \\ &+6\int^t f_0\exp\left[-\int^t\left(a + L\frac{d}{c}\right)\mathrm{d}t\right]\mathrm{d}t, \\ &-\int^t d(t)\exp\left[-3\int^t\left(a + L\frac{d}{c}\right)\mathrm{d}t\right]\mathrm{d}t\bigg). \end{aligned} \tag{35}$$

Example 7. Dai and Jeffrey [13] investigated the following equations

$$\begin{aligned} \alpha(t)u_t - \tfrac{3}{2}a_3(t)uu_x + \tfrac{1}{4}a_3(t)u_{xxx} + \left[\frac{3a_3(t)}{2b_2(t)}b_0(x,t) + \frac{\alpha(t)b_{2t}(t)}{2b_2(t)}x - \tfrac{1}{2}A_2(t)x\right]u_x \\ + \left[\frac{3a_3(t)}{2b_2(t)}b_{0x} + \frac{\alpha(t)b_{2t}(t)}{b_2(t)} - A_2(t)\right]u = 0 \end{aligned} \tag{36}$$

where $b_0(x,t)$ satisfies the equation

$$\alpha(t)b_{0t} + \frac{3a_3(t)}{2b_2(t)}b_0b_{0x} + \tfrac{1}{4}a_3(t)b_{0xxx} + \left[\frac{\alpha(t)b_{2t}}{2b_2(t)} - \tfrac{1}{2}A_2(t)\right]xb_{0x} - A_2(t)b_0 = 0 \tag{37}$$

and

$$\alpha(t)u_t - \tfrac{3}{2}a_3(t)uu_x + \tfrac{1}{4}a_3(t)u_{xxx} - a_1(x,t)u_x - a_{1x}(x,t)u = 0 \tag{38}$$

with

$$\alpha(t)b_{1t} - \frac{3a_3(t)}{8\beta^2}b_1^2b_{1x} + \tfrac{1}{4}a_3(t)b_{1xxx} = 0 \tag{39}$$

and

$$a_1(x,t) = \frac{3a_3(t)}{8\beta^2}b_1^2 + \frac{3a_3(t)}{4\beta}b_{1x} \qquad (\beta = \text{constant}). \tag{40}$$

The inverse scattering transformations for equations (36) and (38) are constructed. Equations (36) and (38) cannot be transformed into the KdV in the view of Dai and Jeffrey. However, we find that such a transformation exists. Let

$$u(x,t) = v(x,t) + \frac{b_0(x,t)}{b_2(t)}. \tag{41}$$

Substituting (41) into (36) and from (37) we obtain

$$\alpha(t)\nu_t - \tfrac{3}{2}a_3(t)\nu\nu_x + \tfrac{1}{4}a_3(t)\nu_{xxx} + \left[\frac{\alpha b_{2t}(t)}{2b_2(t)} - \tfrac{1}{2}A_2(t)\right]x\nu_x + \left[\frac{\alpha b_{2t}(t)}{b_2(t)} - A_2(t)\right]\nu = 0. \tag{42}$$

Let

$$u(x,t) = \nu(x,t) - \frac{2}{3a_3(t)}a_l(x,t) \tag{43}$$

Substituting (43) into (38), we get

$$\alpha(t)\nu_t - \tfrac{3}{2}a_3(t)\nu\nu_x + \tfrac{1}{4}a_3(t)\nu_{xxx} + A(x,t) = 0 \tag{44}$$

where

$$A(x,t) = \alpha(t)\left[\frac{2a_{3t}(t)}{3a_3^2(t)}a_1(x,t) - \frac{2a_{1t}(x,t)}{3a_3(t)}\right] + \frac{2}{3a_3(t)}a_1a_{1x} - \frac{a_{1xxx}}{6}. \tag{45}$$

From equations (39) and (40), we find that $A(x,t) = 0$. Therefore, equations (36) and (38) are essentially the KdV. The equations introduced by Grimshaw [14], Joshi [15], Hlavaty [16], Baby [17] and Lie Yi-shen and Baby [18] are also special cases of equation (1).

3. Transformation from the general m KdV with variable coefficients to the m KdV

We consider the general m KdV equation

$$u_t = f_3(t)u_{xxx} + (c_1 f_2(t)u^p + c_2 e(t)u^{p+1} + c_3 s(t)u^{p-1})u_x + f_0(t)u + f_1(t)xu_x \tag{46}$$

with $p = 1$ and 2, and c_1, c_2, c_3 constants.

Let transformations

$$u(x,t) = g(t)V(\zeta,\tau) \qquad \zeta = xh(t) + l(t) \qquad \tau = m(t) \tag{47}$$

with

$$g(t) = c_0 \exp\left(\int^t f_0(t)\,\mathrm{d}t\right) \qquad h(t) = c_1 \exp\int^t f_1(t)\,\mathrm{d}t \tag{48a}$$

$$l(t) = \int^t c_3\delta_{p1}s(t)h(t)\,\mathrm{d}t \qquad m(t) = \int^t f_3(t)h^3(t)\,\mathrm{d}t. \tag{48b}$$

If the following equations hold

$$f_3(t)h^2(t) = f_2(t)g(t) = e(t)g^2(t) \qquad \text{when } p = 1, c_1c_2 \neq 0 \tag{49a}$$

$$f_3(t)h^2(t) = e(t)g^2(t) \qquad \text{when } p = 1, c_1 = 0 \tag{49b}$$

$$f_3(t)h^2(t) = f_2(t)g^2(t) = e(t)g^3(t) = s(t)g(t) \qquad \text{when } p = 2, c_1c_2c_3 \neq 0 \tag{49c}$$

then equation (46) is transformed into

$$V_\tau = V_{\zeta\zeta\zeta} + c_1VV_\zeta + c_2V^2V_\zeta \qquad \text{when } p = 1 \tag{50}$$

$$V_\tau = V_{\zeta\zeta\zeta} + c_1V^2V_\zeta + c_2V^3V_\zeta + c_3VV_\zeta \qquad \text{when } p = 2. \tag{51}$$

Furthermore, when $c_2 \neq 0$, let $V \to V - c_1/2c_2$, $\zeta \to \zeta - (c_1^2/4c_2)\tau$, $\tau \to \tau$, then the Gardnere equation (50) reads as the m KdV equation

$$V_\tau = V_{\zeta\zeta\zeta} + c_2V^2V_\zeta. \tag{52}$$

When $c_1c_2c_3 \neq 0$ and $c_3 = c_1^2/3c_2$, let $V \to V - c_1/3c_2$, $\zeta \to \zeta + c_1^3/27c_2^2$, $\tau \to \tau$, then equation (51) becomes

$$V_\tau = V_{\zeta\zeta\zeta} + c_2V^3V_\zeta. \tag{53}$$

Evidently, the equation

$$u_t = \nu(t)^3 T_t(t) u_{xxx} + c_1 \nu(t)\omega(t) T_t(t) u u_x + c_2 \omega(t)^2 \nu(t) T_t(t) u^2 u_x + c_3 \nu(t) T_t(t) u_x - \frac{\omega_t(t)}{\omega(t)} u - \frac{\nu_t(t)}{\nu(t)} x u_x \qquad c_1, c_2, c_3 = \text{constants} \tag{54}$$

introduced by Fuchssteiner [1] is a special case of equation (46). By using the transformation

$$u(x,t) = \omega(t)^{-1} V\left(x\nu^{-1}(t) + \frac{4c_2 - c_1^2}{4c_2} T(t), T(t)\right) \tag{55}$$

equation (54) is reduced to equation (52). Pseudopotentials, Lax pairs, BT, ICL, symmetries and Lie algebra for

$$q_t = q_{xxx} - 6q^2 q_x + (F(t)x + G(t))q_x + F(t)q \tag{56}$$

have been given [2, 3, 19]. Evidently, equation (56) can also be reduced to the m KdV, equation (52).

4. Transformation from the general KP equation to the KP equation

The general KP equations with variable coefficients are discussed by many authors, for example, David *et al* [20, 21] discussed a general KP equation with y and t dependence. The Lax pair, BT, solitary wave solution and ICL for the general KP equation

$$u_t = b(t)(6uu_x + u_{xxx}) + k_1(t)(xu_x + 2u) + s_1(t)u_x + [k_2(t)y + s_2(t)]u_y + 6b(t)f(t)u + x(f'(t) - 3k_1(t)f(t) - 12b(t)f^2(t)) + F(t) + 3b(t)g^2(t)D_x^{-1}u_{yy} \tag{57}$$

with $g(t) = \exp\int(2k_1(t) - k_2(t) + 12b(t)f(t))\,dt$ have been given by the author [22]. With $k_2 = s_2 = f = F = 0$, equation (57) reduces to the equation investigated by Gu Zhu-quan [23]. With $b = -1$, $k_i = s_i = 0$, $F(t) = 0$, equation (57) reduces to the equation discussed by Tian Chou [24]. Furthermore, let $f(t) = 1/12t$, $g(t) = \alpha/t$, equation (57) reduces to the Johnson equation, which was discussed for its applications in water of variable depth [25–27]. Painlevé analyses for special cases of equation (57) are also given [28, 29].

Let us now discuss the following general KP equation

$$u_t = f_3(t)u_{xxx} + 6f_2(t)uu_x + f_0(t)u + f_1(t)xu_x + s_1(t)u_x + [k_2(t)y + s_2(t)]u_y + xk_3(t) + F(t) + 3f_4(t)D_x^{-1}u_{yy}. \tag{58}$$

Let $u \to u + \psi(t)$, and $\psi(t) = [\int^t F(t)\exp(-\int^t f_0(t)\,dt) + c_0]\exp(\int^t f_0(t)\,dt)$, then (58) becomes

$$u_t = f_3(t)u_{xxx} + 6f_2(t)uu_x + f_0(t)u + f_1(t)xu_x + [s_1(t) + 6f_2(t)\psi(t)]u_x + [k_2(t)y + s_2(t)]u_y + xk_3(t) + 3f_4(t)D_x^{-1}u_{yy}. \tag{59}$$

Let transformations

$$u(x,t) = x\varphi(t) + g(t)V(\zeta, \eta, \tau) \tag{60a}$$

$$\zeta = p_1(t)x + l(t) \qquad \eta = p_2(t)y + m(t) \qquad \tau = \int^t f_3(t)p_1^3(t)\,dt \tag{60b}$$

$$\varphi'(t) = 6f_2(t)\varphi^2(t) + (f_0(t) + f_1(t))\varphi(t) + k_3(t)$$
$$g(t) = \exp\left[\int^t (6f_2(t)\varphi(t) + f_0(t))\,dt\right] \tag{60c}$$
$$p_1(t) = \exp\left[\int^t (6f_2(t)\varphi(t) + f_1(t))\,dt\right]$$

$$l(t) = \int^t (s_1(t)p_1(t) + 6f_2(t)\psi(t)p_1(t))\,\mathrm{d}t \tag{60d}$$

$$p_2(t) = \exp\left(\int^t k_2(t)\,\mathrm{d}t\right) \qquad m(t) = \int^t s_2(t)p_2(t)\,\mathrm{d}t \tag{60e}$$

if

$$f_3(t)p_1^2(t) = f_2(t)g(t) = f_4(t)p_2^2(t)/p_1^2(t) \tag{61}$$

then equation (58) reduces to the KP equation

$$V_{\zeta\tau} = V_{\zeta\zeta\zeta\zeta} + 6(V_\zeta^2 + VV_{\zeta\zeta}) + 3V_{\eta\eta}. \tag{62}$$

So, the general KP equations with variable coefficients appearing in [22–29] are essentially the KP equation.

We now discuss the KP-type equation

$$u_t = f_3(t)u_{xxx} + 6f_2(t)u^p u_x + f_0(t)u + f_1(t)xu_x + s(t)u_x + f_4(t)D_x^{-1}u_{yy} \qquad p > 0. \tag{63}$$

Let

$$u(x, y, t) = g(t)V(\zeta, \eta, \tau) \qquad \zeta = xh(t) + l(t) \qquad \eta = y \qquad \tau = m(t) \tag{64}$$

where

$$\begin{aligned} g(t) &= c_0 \exp\left(\int^t f_0(t)\,\mathrm{d}t\right) \qquad h(t) = c_1 \exp\left(\int^t f_1(t)\,\mathrm{d}t\right) \\ l(t) &= \int^t s(t)h(t)\,\mathrm{d}t \qquad m(t) = \int^t f_3(t)h^3(t)\,\mathrm{d}t. \end{aligned} \tag{65}$$

If

$$f_3(t)h^2(t) = f_2(t)g^p(t) \qquad f_4(t) = f_3(t)h^4(t) \tag{66}$$

then equation (63) is reduced to the general KP equation

$$V_{\zeta\tau} = V_{\zeta\zeta\zeta\zeta} + 6(V^p V_\zeta)_\zeta + V_{\eta\eta}. \tag{67}$$

So, equation (63) admits the solitary wave solution

$$\begin{aligned} u(x, y, t) &= c_0 \exp\left(\int^t f_0(t)\,\mathrm{d}t\right) \\ &\quad \times \left\{A\,\mathrm{sech}^{2/p}\left[kh(t)x + kl(t) + by - \left(c + \frac{b^2}{k}\right)m(t)\right] + B\delta_{p1}\right\} \end{aligned} \tag{68}$$

where k, b and B are arbitrary constants, A and c are given by (10), $h(t)$, $l(t)$ and $m(t)$ are given by (65), if equation (66) holds.

5. High-order variable coefficients KdV equation

We discuss the high-order variable coefficients KdV equation

$$u_t + c_1 f_1(t)uu_x + \sum_{k=1}^{N} c_{2k+1} f_{2k+1}(t)\frac{\partial^{2k+1}u}{\partial x^{2k+1}} = 0 \qquad c_i = \text{constant}. \tag{69}$$

Let

$$u(x, t) = \varphi(t)x + g(t)V(\zeta, \tau) \qquad \zeta = h(t)x \qquad \tau = \tau(t) \tag{70}$$

where $\varphi(t) = 1/(a_0 + c_1 \int_0^t f_1(t)\,dt)$, $g(t) = h(t) = c_0\varphi(t)$, $\tau(t) = \int_0^t f_{2N+1}(t)h^{2N+1}(t)\,dt$ with a_0, c_0, c_1 arbitrary constants. If

$$f_1(t) = f_3(t)h(t) \qquad f_{2k+1}(t) = f_{2k+3}(t)h^2(t) \qquad (k = 1, 2, 3, \ldots, N-1) \tag{71}$$

then equation (69) is reduced to the high-order KdV equation

$$V_\tau + c_1 V V_\zeta + \sum_{l=1}^{N} c_{2l+1} \frac{\partial^{2l+1} V}{\partial \zeta^{2l+1}} = 0. \tag{72}$$

Let us consider the travelling wave solution for equation (72), $V(\xi, \tau) = W(\eta)$, $\eta = k\xi - \omega\tau$, where k and ω are constants to be determined. Then equation (72) reads

$$-\omega W^{(1)} + c_1 k W W^{(1)} + \sum_{l=1}^{N} c_{2l+1} k^{2l+1} W^{(2l+1)} = 0 \qquad W^{(j)} = \frac{d^j W}{d\eta^j}. \tag{73}$$

Integrating equation (73), we get

$$-\omega W + \tfrac{1}{2} c_1 k W^2 + \sum_{l=1}^{N} c_{2l+1} k^{2l+1} W^{(2l)} = K \tag{74}$$

where K is an integration constant. Further, we assume that the travelling wave solutions to equation (72) are of the particular form

$$W(\eta) = A_0 \operatorname{sech}^{2N}\eta + B_0. \tag{75}$$

Notice that

$$\begin{aligned}(\operatorname{sech}^{2N}\eta)^{(2)} &= -2N(2N+1)\operatorname{sech}^{2N+2}\eta + (2N)^2 \operatorname{sech}^{2N}\eta \\ &\triangleq b_{2,1}\operatorname{sech}^{2N+2}\eta + (2N)^2\operatorname{sech}^{2N}\eta \end{aligned} \tag{76.0}$$

$$\begin{aligned}(\operatorname{sech}^{2N}\eta)^{(4)} &= 2N(2N+1)(2N+2)(2N+3)\operatorname{sech}^{2N+4}\eta \\ &\quad -2N(2N+1)[(2N+2)^2 + (2N)^2]\operatorname{sech}^{2N+2}\eta + (2N)^4\operatorname{sech}^{2N}\eta \\ &\triangleq b_{4,2}\operatorname{sech}^{2N+4}\eta + b_{4,1}\operatorname{sech}^{2N+2}\eta + (2N)^4\operatorname{sech}^{2N}\eta \end{aligned} \tag{76.1}$$

$$(\operatorname{sech}^{2N}\eta)^{(6)} \triangleq b_{6,3}\operatorname{sech}^{2N+6}\eta + b_{6,2}\operatorname{sech}^{2N+4}\eta + b_{6,1}\operatorname{sech}^{2N+2}\eta + (2N)^6\operatorname{sech}^{2N}\eta \tag{76.2}$$

$$\vdots \qquad\qquad \vdots$$

$$\begin{aligned}(\operatorname{sech}^{2N}\eta)^{(2N-2)} &\triangleq b_{2N-2,N-1}\operatorname{sech}^{4N-2}\eta + b_{2N-2,N-2}\operatorname{sech}^{4N-4}\eta + \cdots \\ &\quad + b_{2N-2,1}\operatorname{sech}^{2N+2}\eta + (2N)^{2N-2}\operatorname{sech}^{2N}\eta \end{aligned} \tag{76.$N-1$}$$

$$\begin{aligned}(\operatorname{sech}^{2N}\eta)^{(2N)} &\triangleq (-1)^N 2N(2N+1)(2N+2)\ldots(4N-1)\operatorname{sech}^{4N}\eta \\ &\quad + b_{2N,N-1}\operatorname{sech}^{4N-2}\eta + b_{2N,N-2}\operatorname{sech}^{4N-4}\eta + \cdots \\ &\quad + b_{2N,1}\operatorname{sech}^{2N+2}\eta + (2N)^{2N}\operatorname{sech}^{2N}\eta. \end{aligned} \tag{76.N}$$

Substituting equations (75) and (76) into equation (74), we get the following equations:

$$\operatorname{sech}^{2N}\eta : \omega = c_1 k B_0 + (2N)^2 c_3 k^3 + (2N)^4 c_5 k^5 + \cdots + (2N)^{2N} c_{2N+1} k^{2N+1} \tag{77.0}$$

$$\operatorname{sech}^{2N+2}\eta : b_{2,1}c_3 + b_{4,1}c_5 k^2 + b_{6,1}c_7 k^4 + \cdots + b_{2N,1}c_{2N+1}k^{2N-2} = 0 \tag{77.1}$$

$$\operatorname{sech}^{2N+4}\eta : b_{4,2}c_5 + b_{6,2}c_7k^2 + \cdots + b_{2N,2}c_{2N+1}k^{2N-4} = 0 \qquad (77.2)$$

$$\vdots \qquad\qquad \vdots$$

$$\operatorname{sech}^{4N-2}\eta : b_{2N-2,N-1}c_{2N-1}k^{2N-1} + b_{2N,N-1}c_{2N+1}k^{2N+1} = 0 \qquad (77.N-1)$$

$$\operatorname{sech}^{4N}\eta : A_0 = \frac{(-1)^{N+1}}{c_1}(4N)(2N+1)(2N+2)\ldots(4N-1)c_{2N+1}k^{2N}. \qquad (77.N)$$

From equation $(77.N-1)$, assuming $c_{2N-1}c_{2N+1} < 0$ and noticing that $b_{2N-2,N-1}b_{2N,N-1} > 0$, we get

$$k^2 = -\frac{b_{2N-2,N-1}c_{2N-1}}{b_{2N,N-1}c_{2N+1}} > 0. \qquad (78)$$

From equations $(77.N-2)$–(77.1), we obtain

$$\begin{aligned} c_{2N-3} &= d_{2N-3}\frac{c_{2N-1}^2}{c_{2N+1}} \qquad & c_{2N-5} &= d_{2N-5}\frac{c_{2N-1}^3}{c_{2N+1}^2}\cdots \\ c_5 &= d_5\frac{(c_{2N-1})^{N-2}}{(c_{2N+1})^{N-3}} \qquad & c_3 &= d_3\frac{(c_{2N-1})^{N-1}}{(c_{2N+1})^{N-2}} \end{aligned} \qquad (79)$$

where d_i are dependent on N.

So, if $c_{2N-1}c_{2N+1} < 0$, $c_3, c_5, \ldots, c_{2N+1}$ satisfy equation (79), then equation (72) admits the solitary wave solution

$$V(\xi,\tau) = A_0\operatorname{sech}^{2N}(k\xi - \omega\tau) + B_0 \qquad (80)$$

where B_0 is an arbitrary constant and A_0, ω and k are given by equations $(77.N)$, (77.0) and (78). We thus get the solution for equation (69)

$$\begin{aligned} u(x,t) = {} & \frac{x}{a_0 + c_1\int_0^t f_1(t)\,dt} + \frac{c_0}{a_0 + c_1\int_0^t f_1(t)\,dt} \\ & \times\left\{A_0\operatorname{sech}^{2N}\left[k\left(\frac{c_0x}{a_0 + c_1\int_0^t f_1(t)\,dt}\right.\right.\right. \\ & \left.\left.\left. -c\int_0^t \frac{c_0^{2N+1}f_{2N+1}(t)}{(a_0 + c_1\int_0^t f_1(t)\,dt)^{2N+1}}\,dt\right)\right] + B_0\right\}. \end{aligned} \qquad (81)$$

6. Discussion

In the previous sections, a class of explicit transformations between various variable coefficient equations of KdV, *m*KdV and KP type to their (integrable) constant coefficient counterparts are revealed. Therefore, integrability and symmetry results, for instance, PP, BT, Lax pairs, solitary wave solutions, ICL, symmetry, Lie algebra etc for variable coefficients KdV, *m*KdV and KP-type equations are simple, transparent and straightforward. So, it is very important and interesting to find new and real KdV-type equations with variable coefficients, and to investigate integrability and symmetries for those equations. Of course, a few equations with variable coefficients which cannot be reduced to the standard forms have been introduced: see, for instance, the KP equation with explicit x and t dependence introduced by Steeb and Spieker (see equation (17a) in [30]), the general KP equation with explicit x, y and t dependence and the general x, t-dependent KdV–Burgers-type equation studied by Zhu zuo-nong (see equations (26) and (32) in [31]), the equation

$$u_t = \nu^3(t)T_t(t)u_{xxx} + 6\nu(t)w(t)T_t(t)uu_x - \left[\frac{1}{T(t)} + \frac{w_t(t)}{w(t)}\right]u - \left[\frac{1}{T(t)} + \frac{\nu_t(t)}{\nu(t)}\right]xu_x \tag{82}$$

discussed by Fuchssteiner (see equation (3.74) in [1]), and the model based on the forced KdV equation

$$u_t - 6uu_x + u_{xxx} = A\sin[q(x - \nu t)]. \tag{83}$$

This equation has recently been analysed by Malomed [32] and Grimshaw *et al* [33]. Winternitz and Gazeau [34, 35] studied the symmetry for

$$u_t + f(x,t)uu_x + g(x,t)u_{xxx} = 0. \tag{84}$$

It appears to the author that no simple transformation exists which transforms equations (82) or (84) into a constant coefficient equation. However, if $T(t)$ satisfies

$$c_1 \ln(c_1 - T) + T = -t + c_2 \tag{85}$$

$u \to u + x/6\nu wT_tT$, then equation (82) is reduced to the KdV. If $f_x = g_x = 0$, $g(t) = f(t)[c_1 \int_0^t f(s)\,ds + c_2]$, then equation (84) is also reduced to the KdV.

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References

[1] Fuchssteiner B 1993 *J. Math. Phys.* **34** 5140
[2] Lou Sen-yue 1991 *J. Phys. A: Math. Gen.* **24** L513
[3] Lou Sen-yue 1992 *Acta Phys. Sin.* **41** 182
[4] Lie Yi-shen and Zhu G C 1986 *Chin. Sci. Bull.* **19** 1449
[5] Nirmala N, Veden M J and Baby B V 1986 *J. Math. Phys.* **27** 2640
[6] Brugarino T 1989 *J. Math. Phys.* **30** 1013
[7] Hirota R and Satsuma J 1976 *J. Phys. Soc. Japan. Lett.* **41** 2141
[8] Maxon S and Viecelli J 1974 *Phys. Fluids* **17** 1614
[9] Hirota R 1979 *J. Phys. Soc. Japan Lett.* **46** 1681
[10] Lou Sen-yue 1991 *Sci. Chin.* A **6** 662
[11] Tian Chou 1987 *J. Phys. A: Math. Gen.* **20** 359
[12] Zhu Zuo-nong 1992 *Acta Phys. Sin.* **41** 409
[13] Dai H H and Jeffrey A 1989 *Phys. Lett.* **139A** 369
[14] Grimshaw R 1979 *Proc. R. Soc.* A **368** 359
[15] Joshi N 1987 *Phys. Lett.* **125A** 456
[16] Hlavaty L 1988 *Phys. Lett.* **128A** 335
[17] Baby B V 1986 *J. Phys. A: Math. Gen.* **20** L555
[18] Lie Yi-shen and Baby B V 1986 Variable coefficient Korteneg de Vries equation: Darboux's transformation, exact solutions, and conservations laws *International Center for Theoretical Physics, Preprint* IC/86/340; 1986 Symmetries of KdV-type equation *Preprint* IC/86/398
[19] Tian Chou 1987 *J. Phys. A: Math. Gen.* **20** 367
[20] David D, Levi D and Winternitz P 1987 *Stud. Appl. Math.* **76** 133
[21] David D, Levi D and Winternitz P 1989 *Stud. Appl. Math.* **80** 1

[22] Zhu Zuo-nong 1993 *Phys. Lett.* **180A** 409
[23] Gu Zhu-quan 1989 *Chin. Sci. Bull.* 2 86
[24] Tian Chou 1988 *Chin. Sci. Bull.* 2 84
[25] Johnson R S 1983 *Nonlinear Waves* ed L Debnath (Cambridge: Cambridge University Press)
[26] Debnath L 1985 *Advances in Nonlinear Waves* (Boston: Pitman)
[27] Dryuma V S 1983 *Sov. Math. Dokl.* 27 6
[28] Zhu Zuo-nong 1993 *Phys. Lett.* **182A** 277
[29] Oevel W and Steeb W-H 1984 *Phys. Lett.* **103A** 239
[30] Steeb W-H and Spieker B-M 1985 *Phys. Rev.* A 31 1952
[31] Zhu Zuo-nong 1994 *Phys. Lett.* **185A** 287
[32] Malomed B A 1992 *Phys. Lett.* **172A** 373
[33] Grimshaw R, Malomed B A and Xin Tian 1993 *Phys. Lett.* **179A** 291
[34] Winternitz P and Gazeau J P 1992 *Phys. Lett.* **167A** 246
[35] Gazeau J P and Winternitz P 1992 *J. Math. Phys.* 33 4087